Theoretical Computer Science 420 (2012) 56–63

Contents lists available at SciVerse ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Decidability, undecidability, and PSPACE-completeness of the twins property in the tropical semiring

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ARTICLE INFO

Article history:
Received 9 September 2010
Received in revised form 27 October 2011
Accepted 4 November 2011
Communicated by D. Perrin

Keywords:
Weighted finite automata
Formal power series
Tropical semiring
Twins property

ABSTRACT

We solve a problem already investigated by Mohri in 1997: we show that the twins property for weighted finite automata over the tropical semiring is decidable and PSPACE-complete. We also point out that it is undecidable whether two given states are twins.

1. Introduction

Weighted finite automata over the tropical semiring (WFA) are studied under various names in the literature, e.g. distance, finance, or cost automata. They have also appeared in various contexts: logical problems in formal language theory (star height, finite power property, star problem for traces) [12,15,16,24,28], study of dynamics of some discrete event systems (DES) [9,10], automatic speech recognition [25], and database theory [11].

To achieve efficient implementations, one is interested in deterministic (sequential) WFA [25] which rises the determinization (sequentiality) problem: decide (constructively) whether some given min-plus automaton admits a deterministic equivalent. Its decidability was shown in 2004 by Klimann, Lombardy, Mairesse, and Prieur for finitely ambiguous WFA [20] and recently by Lombardy and the author for polynomially ambiguous (cycle unambiguous) WFA [18]. Despite this progress, the determinization problem is considered as wide open.

In 1997, Mohri presented an imperfect algorithm [25,26]. The algorithm tries to construct some deterministic equivalent by a generalization of the power set construction. If the algorithm terminates on some given WFA $\mathcal{A}$, then it constructs a deterministic equivalent. For many WFA, in particular for many WFA in practical applications, the algorithm successfully constructs a deterministic equivalent. However, the existence of a deterministic equivalent is necessary but not sufficient for the termination, and for many WFA, the algorithm does not terminate even if a deterministic equivalent exists.

To study the termination of his algorithm, Mohri adapted the notion of the twins property from [7] which is a sufficient condition for the termination of his algorithm [25,26]. Mohri's algorithm and the notion of the twins property became a popular object, see e.g. [1,17,19,26] or [5,6,21] for a generalization to trees.

For cycle unambiguous WFA, the twins property is decidable in polynomial time [1,2,25,26]. In general, the decidability of the twins property remained open. It was rather conjectured that the *un*decidability of the twins property follows easily from a result by Krob from 1994 which says that the semantic equivalence of two given WFA is undecidable [22]. Indeed, we will observe that a problem closely related to the twins property (to decide whether two given states are twins) is undecidable.

As our main result, we will show that the twins property is decidable and PSPACE-complete.

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0304-3975/$ – see front matter
doi:10.1016/j.tcs.2011.11.006

2. Overview

2.1. Some definitions

To give some more background, we need precise definitions. See [3,4,8,23,27] for more information.

Let $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ and $\mathbb{Z}_\infty = \mathbb{Z} \cup \{\infty\}$. We call $\mathbb{Z}_\infty$ equipped with minimum and addition the *tropical semiring*. We denote multiplication of matrices over $\mathbb{Z}_\infty$ by $\cdot$ or juxtaposition.

Let us recall some facts on matrix multiplication in the tropical semiring. Let Q be a finite set, $n \geq 1$, and $M_1, \ldots, M_n \in \mathbb{Z}_\infty^{Q\times Q}$. A sequence $p_0, \ldots, p_n$ is called *victorious* (for $M_1, \ldots, M_n$) if

$$M_1[p_0, p_1] + \cdots + M_n[p_{n-1}, p_n] \in \mathbb{Z} \tag{1}$$

and for every $p'_0, \ldots, p'_n$ satisfying $p'_0 = p_0$ and $p'_n = p_n$, we have

$$M_1[p_0, p_1] + \cdots + M_n[p_{n-1}, p_n] \leq M_1[p'_0, p'_1] + \cdots + M_n[p'_{n-1}, p'_n].$$

We call the sum in (1) *the sum along* $p_0, \ldots, p_n$ (for $M_1, \ldots, M_n$).

Let $p_0, \ldots, p_n$ be a victorious sequence for $M_1, \ldots, M_n$ and let $1 \leq k \leq \ell \leq n$. The subsequence $p_{k-1}, \ldots, p_\ell$ is a victorious sequence for $M_k, \ldots, M_\ell$. Indeed, if you assume a counter example for the minimality criterion for $p_{k-1}, \ldots, p_\ell$, then you can replace in $p_0, \ldots, p_n$ the states $p_{k-1}, \ldots, p_\ell$ by this counter example and contradict the minimality criterion for $p_0, \ldots, p_n$. For every $p, q \in Q$ satisfying $(M_1 \cdots M_n)[p, q] \in \mathbb{Z}$ there is a victorious sequence $p = p_0, \ldots, p_n = q$.

For every victorious sequence $p_0, \ldots, p_n$, the sum along $p_0, \ldots, p_n$ yields $(M_1 \cdots M_n)[p_0, p_n]$.

2.2. Weighted finite automata over the tropical semiring

Throughout the paper, let Σ be some finite alphabet. For $w \in \Sigma^*$, let $|w|$ be the length of w.

A *weighted finite automaton over the tropical semiring (WFA)* is a tuple $\mathcal{A} = [Q, \lambda, \mu, \varrho]$ where Q is a finite set called *states*, $\lambda, \varrho \in \mathbb{Z}_\infty^Q$, and $\mu : \Sigma^* \to \mathbb{Z}_\infty^{Q\times Q}$ is a homomorphism into the semiring of $Q \times Q$-matrices over $\mathbb{Z}_\infty$

A WFA $\mathcal{A}$ defines a mapping $|\mathcal{A}| : \Sigma^* \to \mathbb{Z}_\infty$ by $|\mathcal{A}|(w) = \lambda \cdot \mu(w) \cdot \varrho$ for $w \in \Sigma^*$. Given WFA $|\mathcal{A}_1|$, $|\mathcal{A}_2|$, it is undecidable whether $|\mathcal{A}_1| = |\mathcal{A}_2|$ (Krob's theorem, [22]).

Let $\mathcal{A} = [Q, \lambda, \mu, \varrho]$ be a WFA. We call some $q \in Q$ an *initial state* if $\lambda[q] \in \mathbb{Z}$.

We freely assume that $\mathcal{A}$ is *trim*, i.e. for every $q \in Q$, there are $u, v \in \Sigma^*$ such that $(\lambda \cdot \mu(u))[q] \in \mathbb{Z}$ and $(\mu(v) \cdot \varrho)[q] \in \mathbb{Z}$.

If $\mathcal{A}$ has exactly one initial state and for every $a \in \Sigma$, $p \in Q$, there is at most one $q \in Q$ satisfying $\mu(a)[p, q] \in \mathbb{Z}$, we call $\mathcal{A}$ *deterministic*.

Given some $w = a_1 \ldots a_{|w|} \in \Sigma^*$, we call $q_0, \ldots, q_{|w|}$ a *path* for w if

$$\lambda[q_0] + \mu(a_1)[q_0, q_1] + \cdots + \mu(a_{|w|})[q_{|w|-1}, q_{|w|}] + \varrho[q_{|w|}] \in \mathbb{Z}.$$

We call $\mathcal{A}$ *unambiguous* if there is at most one path for every word w. If there exists a polynomial P such that for every $w \in \Sigma^*$, there are at most $P(|w|)$ paths, then we call $\mathcal{A}$ *polynomially* ambiguous.

If for every $w = a_1 \ldots a_{|w|} \in \Sigma^*$ and every $q \in Q$, there is at most one sequence $q = q_0, \ldots, q_{|w|} = q$ satisfying $\mu(a_1)[q_0, q_1] + \cdots + \mu(a_{|w|})[q_{|w|-1}, q_{|w|}] \in \mathbb{Z}$, then we call $\mathcal{A}$ *cycle unambiguous*.

Every deterministic WFA is unambiguous and every unambiguous WFA is polynomially[1] ambiguous. A WFA is polynomially ambiguous iff it is cycle unambiguous [13,14]. The classes of mappings computable by deterministic, unambiguous resp. polynomially ambiguous WFA form a strict hierarchy [17,20].

For a description of Mohri's algorithm the reader is referred to [25,26,19]. For polynomially ambiguous WFA, it is decidable whether Mohri's algorithm terminates [17], and moreover, the determinization problem is decidable [18].

To study the termination of his algorithm, Mohri adapted the notion of the twins property from [7]: Two states $p, q \in Q$ are called *siblings* if there exist some $w \in \Sigma^*$ such that

$$(\lambda \cdot \mu(w))[p] \in \mathbb{Z} \quad \text{and} \quad (\lambda \cdot \mu(w))[q] \in \mathbb{Z}.$$

Two states $p, q \in Q$ are called *twins* [25,26] if for every word satisfying

$$\mu(w)[p, p] \in \mathbb{Z} \quad \text{and} \quad \mu(w)[q, q] \in \mathbb{Z}, \quad \text{we have } \mu(w)[p, p] = \mu(w)[q, q].$$

A WFA $\mathcal{A}$ has the *twins property* iff every siblings are twins. Mohri showed in 1997 that the twins property is a sufficient (but not a necessary) condition for the termination of his algorithm [25,26].

Despite their disadvantages, Mohri's algorithm and the notion of the twins property became a popular object, see e.g. [1,17,19,26] or [5,6,21] for a generalization to trees.

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[1] The number of paths is bounded by the polynomial $P(x) = 1$ for $x \in \mathbb{N}$.

2.3. Main results

For cycle unambiguous WFA, the twins property is decidable in polynomial time [1,2,25,26]. To decide the twins property, one is mainly interested in values $\mu(w)[q, q]$ for $w \in \Sigma^*$, $q \in Q$. If the input WFA is cycle unambiguous, then these values are determined unambiguously, i.e., the twins property is essentially an equivalence problem for unambiguous WFA and rather easy to decide. The known decision methods work by deciding for every pair p, q of siblings whether p and q are twins.

For arbitrary WFA, the situation is entirely different. The values $\mu(w)[q, q]$ are not determined unambiguously. The attempt to generalize the known methods from cycle unambiguous to arbitrary WFA leads us to the equivalence problem for WFA which is undecidable by Krob's theorem [22]. Indeed, it is easy to conclude from Krob's theorem that for given siblings p, q, it is undecidable whether p and q are twins:

Theorem 1. *Given a WFA $\mathcal{A}$ and two siblings q_1 and q_2, it is undecidable whether q_1 and q_2 are twins.*

Theorem 1 seems to be folklore, we included it for the sake of completeness (see Section 3). A widespread misbelief is closely related to Theorem 1: as a conclusion from Krob's theorem the twins property is undecidable for WFA over the tropical semiring. However, to decide the twins property means to decide whether *every* pair of siblings are twins rather than deciding whether a *given* pair of siblings are twins. We clear out this misbelief and show that the twins property is decidable. In fact, we even show that the twins property is decidable in PSPACE (Section 4). We also show that the twins property is PSPACE-hard (Section 3). To sum up:

Theorem 2. *Deciding the twins property of a given WFA is PSPACE-complete.*

3. Undecidability and PSPACE-hardness

In this section, we deal with the rather negative results. Theorem 1 seems to be folklore. It has a crucial impact on our strategy to decide the twins property.

Proof (*Theorem* 1)**.** We assume the contrary and show that the equivalence problem for WFA is decidable, contradicting Krob's result.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be WFA. We want to decide whether $|\mathcal{A}_1| = |\mathcal{A}_2|$.

At first, we can construct for $i \in \{1, 2\}$ an automaton recognizing the support $\mathsf{supp}(|\mathcal{A}_i|) = \{w \in \Sigma^* \mid |\mathcal{A}_i|(w) \in \mathbb{Z}\}$ and decide whether $\mathsf{supp}(|\mathcal{A}_1|) = \mathsf{supp}(|\mathcal{A}_2|)$. If the supports differ, we are done. Otherwise, we proceed the following construction: let $\# \notin \Sigma$. By standard techniques, we can construct a WFA $\mathcal{A}$ with sibling states q_1, q_2 which satisfies the following properties for $i \in \{1, 2\}$:

1. For every $n \geq 0$, $w_1, \ldots, w_n \in \Sigma^*$, we have
$$\mu\big(\#w_1\#w_2\ldots\#w_n\big)[q_i, q_i] = \sum_{\ell=1}^{n} |\mathcal{A}_i|(w_\ell).$$
2. For every $w \in \Sigma(\Sigma \cup \#)^*$, we have $\mu(w)[q_i, q_i] = \infty$.

If $|\mathcal{A}_1| = |\mathcal{A}_2|$, then q_1, q_2 are twins. Otherwise, there is some $w \in \Sigma^*$ such that $|\mathcal{A}_1|(w) \neq |\mathcal{A}_2|(w)$ but $|\mathcal{A}_1|(w),\ |\mathcal{A}_2|(w) \in \mathbb{Z}$, and hence,

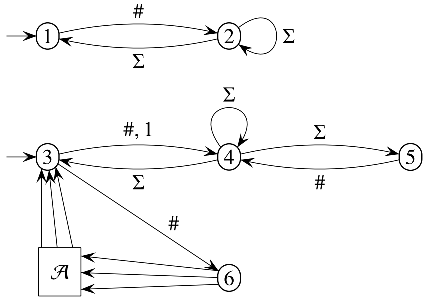
$$\mu(\#w)[q_1, q_1] = |\mathcal{A}_1|(w) \neq |\mathcal{A}_2|(w) = \mu(\#w)[q_2, q_2],$$

i.e., q_1 and q_2 are not twins. To sum up, q_1, q_2 are twins iff $|\mathcal{A}_1| = |\mathcal{A}_2|$, i.e., by deciding whether q_1, q_2 are twins, we can decide whether $|\mathcal{A}_1| = |\mathcal{A}_2|$. $\square$

By a similar reduction, we show that the twins property is PSPACE-hard.

Proof (*PSPACE-Hardness in Theorem* 2)**.** Let $\mathcal{A}$ be a usual (unweighted) automaton, i.e., assume $\mathcal{A} = [Q, E, I, F]$ whereas Q is a finite set, $E \subseteq Q \times \Sigma \times Q$, $I, F \subseteq Q$. Then, $\mathcal{A}$ just accepts some language $L(\mathcal{A}) \subseteq \Sigma^*$. To decide whether $L(\mathcal{A}) = \Sigma^+$ is well-known PSPACE-complete, and hence, PSPACE-hard.

We construct from $\mathcal{A}$ a WFA $\mathcal{A}' = [Q', \lambda', \mu', \varrho']$ such that $\mathcal{A}'$ satisfies the twins property iff $L(\mathcal{A}) = \Sigma^+$. Moreover, $\mathcal{A}'$ has just 6 states more than $\mathcal{A}$. Again, let $\# \notin \Sigma$. The construction of $\mathcal{A}' = [Q', \lambda', \mu', \varrho']$ is shown in the following picture:



The edge from state 3 to 4 with the label #, 1 means that $\mu'(\#)[3, 4] = 1$. The edge from state 1 to 2 with the label # means that $\mu'(\#)[1, 2] = 0$. The edge from 2 to 1 with label Σ means that for every $a \in \Sigma$, we have $\mu(a)[2, 1] = 0$.

There is no edge from state 1 to 4. This means that $\mu'(x)[1, 4] = \infty$ for every $x \in \# \cup \Sigma$.

The box with label $\mathcal{A}$ indicates a copy of $\mathcal{A}$. For every word $w \in L(\mathcal{A})$, we have $\mu'(w)[6, 3] = 0$. For every $w \in \Sigma^+ \setminus L(\mathcal{A})$, we have $\mu'(w)[6, 3] = \infty$. The states of $\mathcal{A}'$ are $1, \ldots, 6$ and the states of $\mathcal{A}$.

The incoming arrows at states 1 and 3 indicate that 1 and 3 are initial states, i.e., $\lambda'[1] = \lambda'[3] = 0$ but $\lambda'[q] = \infty$ for any other state q. We did not specify ϱ' because ϱ' is not relevant for the twins property.

The key idea is that $\mathcal{A}'$ can read words of the form $(\#\Sigma^+)^*$ from state 1 to state 1 as well as from state 3 to state 3. The key difference is that when reading from state 1 to state 1, the weight is always 0, but when reading from 3 to 3 the weight is larger than 0 if the factors between the #'s are not accepted by $\mathcal{A}$.

Let us remark the importance of state 5. For now, assume that state 5 does not exist. Let $a \in \Sigma$. Clearly, 2 and 4 are siblings: $\mu'(\#)[1, 2] = 0$ and $\mu'(\#)[3, 4] = 1$. However, regardless of whether $L(\mathcal{A}) = \Sigma^+$, the states 2 and 4 are not twins: $\mu'(a\#)[2, 2] = 0$ but $\mu'(a\#)[4, 4] = 1$. Consequently, $\mathcal{A}'$ does not satisfy the twins property.

To complete the proof, it remains to show that $\mathcal{A}'$ satisfies the twins property iff $L(\mathcal{A}) = \Sigma^+$.

Assume that $L(\mathcal{A}) \neq \Sigma^+$, i.e., there is some $w \in \Sigma^+$ such that $w \notin L(\mathcal{A})$. Clearly, 1 and 3 are siblings but $\mu'(\#w)[1, 1] = 0$ and $\mu'(\#w)[3, 3] = 1$, i.e., $\mathcal{A}'$ does not satisfy the twins property.

Conversely, assume $L(\mathcal{A}) = \Sigma^+$. We show that for every state q and every $w \in (\# \cup \Sigma)^*$, we have $\mu'(w)[q, q] \in \{0, \infty\}$, and consequently, $\mathcal{A}'$ has the twins property. Let $w \in (\# \cup \Sigma)^*$. We distinguish various cases on w.

Case 1: $w \in \Sigma^*$

By construction, all transitions in $\mathcal{A}'$ which are labeled with letters from Σ are weighted with 0. Hence, $\mu'(w)[q, q] \in \{0, \infty\}$ for every state q.

Case 2: $w \in (\# \cup \Sigma)^*\#\#(\# \cup \Sigma)^*$

By construction, $\mathcal{A}'$ cannot read words with two consecutive #'s, and hence, $\mu'(w)[q, q] = \infty$ for every state q.

Case 3: $w \in (\# \cup \Sigma)^*\#(\# \cup \Sigma)^*$, but $w \notin (\# \cup \Sigma)^*\#\#(\# \cup \Sigma)^*$

Case 3.1: $w \in \#(\# \cup \Sigma)^*$

For $q \in \{2, 4, 6\}$, or for states q inside $\mathcal{A}$, we have $\mu'(w)[q, q] = \infty$. For $q \in \{1, 3, 5\}$,[2] we have $\mu'(w)[q, q] = 0$ or $\mu'(w)[q, q] = \infty$ depending on the last letter of w.

Case 3.2: $w \in \Sigma(\# \cup \Sigma)^*$

For $q \in \{1, 3, 5\}$, we have $\mu'(w)[q, q] = \infty$. For $q \in \{2, 4\}$, we have $\mu'(w)[q, q] = 0$.

Assume that $q = 6$ or q is a state inside $\mathcal{A}'$. Assume further $\mu'(w)[q, q] \neq \infty$. Let w_1 be the prefix of w up to the first #, let w_3 be shortest the suffix of w starting with #, and let w_2 be "the rest in the middle" of w. More precisely, we can write $w = w_1 w_2 w_3$ such that $w_1 \in \Sigma^+$, $w_2 \in (\#\Sigma^+)^*$, and $w_3 \in \#\Sigma^*$.

Since $\mu'(w)[q, q] \neq \infty$, $\mathcal{A}'$ can read w from q to q. Imagine how $\mathcal{A}'$ reads w from q. It cannot read w entirely inside $\mathcal{A}$, since $\mathcal{A}$ cannot read #'s. To leave $\mathcal{A}$ via state 3, we need a #. To reenter $\mathcal{A}$ to arrive at q, we need a # to get to state 6. Consequently, the only way to read w from q to q is to read w_1 inside $\mathcal{A}$ and to arrive at 3 and to read w_3 from state 3 to q.

Now, we take care on w_2. Since $w_2 \in (\#\Sigma)^*$ and $L(\mathcal{A}) = \Sigma^+$, $\mathcal{A}'$ can read w_2 from state 3 to state 3 by reading each factor of the form $\#\Sigma^+$ via state 6 and $\mathcal{A}$.

To sum up, $\mathcal{A}$ can read w_1 from q to 3, w_2 from 3 to 3, and w_3 from 3 to q in a way that only transition weighted with 0 are used. Hence, our above assumption $\mu'(w)[q, q] \neq \infty$ implies $\mu'(w)[q, q] = 0$, i.e., $\mu'(w)[q, q] \in \{0, \infty\}$. □

4. Decidability

Let $\mathcal{A} = [Q, \lambda, \mu, \varrho]$ be a WFA. It is easy to establish an algorithm which searches for a counter example for the twins property. This algorithm terminates iff $\mathcal{A}$ does not satisfy the twins property. To get a decision algorithm, we improve the search method by compressing the search space. If $\mathcal{A}$ satisfies the twins property, then the search space is compressed to a finite size. Otherwise, the compressed search space might be infinite, but the algorithm will find the counter example.

We present two decision algorithms which rely on the same idea. The first algorithm is rather practical. The second algorithm is non-deterministic, designed towards low space-complexity, and hence, very theoretical (proof of Theorem 2).

We denote by $\mathrm{dg}(\mathcal{A})$ the difference between the biggest and the least integer of the set $\{\mu(a)[p, q] \mid a \in \Sigma, p, q \in Q\}$.[3]

Lemma 1 and its proof are adapted from [25] where Mohri proves that the twins property is a sufficient condition for the termination of his algorithm. It allows us to prove finiteness of the compressed search space.

2 In the case $q = 3$ we need the assumption $L(\mathcal{A}) = \Sigma^+$.

3 We set $\mathrm{dg}(\mathcal{A}) = 0$ if this set is empty.

Lemma 1. *Assume that $\mathcal{A}$ has the twins property and let p, q be siblings of $\mathcal{A}$. For every $w \in \Sigma^*$ and every $p', q' \in Q$ satisfying $\mu(w)[p, p'],\ \mu(w)[q, q'] \in \mathbb{Z}$, we have*

$$\left|\mu(w)[p, p'] - \mu(w)[q, q']\right| \;\leq\; |Q|^2 \cdot dg\,(\mathcal{A}).$$

Proof. Let p, q, p', q' and w as in the lemma. Denote $w = a_1 \dots a_{|w|}$. Let $p = p_0, \dots, p_{|w|} = p'$ and $q = q_0, \dots, q_{|w|} = q'$ be victorious sequences for $\mu(a_1), \dots, \mu(a_{|w|})$.

For $0 \leq i \leq |w|$, the states p_i and q_i are siblings.

Let $0 \leq i < j \leq |w|$ satisfying $p_i = p_j$ and $q_i = q_j$. By the twins property, we have

$$\mu(a_{i+1} \dots a_j)[p_i, p_j] \;=\; \mu(a_{i+1} \dots a_j)[q_i, q_j]. \tag{2}$$

Now, $p_i, \dots, p_j$ and $q_i, \dots, q_j$ are victorious, and by (2) the sums along $p_i, \dots, p_j$ and $q_i, \dots, q_j$ for $\mu(a_{i+1}), \dots, \mu(a_j)$ coincide.

Let i_1 be the least integer such that the pair (p_{i1}, q_{i1}) occurs twice in $(p_0, q_0), \dots, (p_{|w|}, q_{|w|})$. Let j_1 be the largest integer satisfying $(p_{i1}, q_{i1}) = (p_{j1}, q_{j1})$. We introduce i_2, j_2 by the same principle for the rest list $(p_{j1}, q_{j1}), \dots, (p_{|w|}, q_{|w|})$ and so on until we achieve a rest list which is repetition free. In this way, we get some k and $0 \leq i_1 < j_1 < i_2 < \dots < j_k \leq |w|$.

Now, we can prove the lemma: to examine the difference between $\mu(w)[p, p']$ and $\mu(w)[q, q']$, we compare the sums along $p_0, \dots, p_{|w|}$ and along $q_0, \dots, q_{|w|}$. As seen above, the sums along the sequences $p_{i1}, \dots, p_{j1}$ and $q_{i1}, \dots, q_{j1}$ coincide. The same coincidence holds for any cycle from i_ℓ to j_ℓ for some $1 \leq \ell \leq k$. Hence, the sums along $p_0, \dots, p_{|w|}$ and $q_0, \dots, q_{|w|}$ coincide up to the summands which are not part of some $i_\ell j_\ell$-cycle. By the choice of $i_1, j_1, \dots, i_{|w|}, j_{|w|}$, the number of these summands is at most $|Q|^2$ and the claim follows. □

We denote by ∞^Q the tuple $(\infty, \dots, \infty) \in \mathbb{Z}_\infty^Q$.

Given $m \in \mathbb{Z}_\infty$, $x \in \mathbb{Z}_\infty^Q$, we define $m \oplus x \in \mathbb{Z}_\infty^Q$ by $(m \oplus x)[q] = m + x[q]$. If we consider m as a 1×1-matrix and x as a row-matrix, then $m \oplus x$ is the matrix product in the tropical semiring. Consequently, we have for $m \in \mathbb{Z}_\infty$, $x \in \mathbb{Z}_\infty^Q$, $M \in \mathbb{Z}_\infty^{Q \times Q}$ the associativity law $(m \oplus x) \cdot M \;=\; m \oplus (x \cdot M)$.

Given $x, y \in \mathbb{Z}_\infty^Q$, we define the normalization of the pair (x, y) denoted by $[\![(x, y)]\!]$. If $x = y = \infty^Q$, then let $[\![(x, y)]\!] = (\infty^Q, \infty^Q)$. Otherwise, we set $m = \mathrm{Min}\{x[q], y[q] \mid q \in Q\}$ and $[\![(x, y)]\!] = \big((-m) \oplus x,\ (-m) \oplus y\big)$.

Remark 1. *For every $m \in \mathbb{Z}$, $x, y \in \mathbb{Z}_\infty^Q$, we have $[\![(m \oplus x, m \oplus y)]\!] = [\![(x, y)]\!]$.*

Now, we define a mapping $(\mathbb{Z}_\infty^Q \times \mathbb{Z}_\infty^Q) \times \Sigma^* \to \mathbb{Z}_\infty^Q \times \mathbb{Z}_\infty^Q$. For every $x, y \in \mathbb{Z}_\infty^Q,\ w \in \Sigma^*$, we set

$$(x, y) \cdot w = \Big[\!\!\Big[\big(x \cdot \mu(w),\ y \cdot \mu(w)\big)\Big]\!\!\Big].$$

Lemma 2. *For every $x, y \in \mathbb{Z}_\infty^Q,\ u, v \in \Sigma^*$, we have*

$$((x, y) \cdot u) \cdot v = (x, y) \cdot (uv).$$

Proof. By definition,

$$((x, y) \cdot u) \cdot v = \Big[\!\!\Big[\big(x \cdot \mu(u),\ y \cdot \mu(u)\big)\Big]\!\!\Big] \cdot v.$$

There is some $m \in \mathbb{Z}$ such that[4]

$$\dots = \big(m \oplus x \cdot \mu(u),\ m \oplus y \cdot \mu(u)\big) \cdot v$$

which is by definition

$$\dots = \Big[\!\!\Big[\big(m \oplus x \cdot \mu(u) \cdot \mu(v),\ m \oplus y \cdot \mu(u) \cdot \mu(v)\big)\Big]\!\!\Big].$$

We use that μ is a homomorphism, apply Remark 1 and get

$$\dots = \Big[\!\!\Big[\big(x \cdot \mu(uv),\ y \cdot \mu(uv)\big)\Big]\!\!\Big],$$

which is by definition the right hand side. □

For $q \in Q$ let $e_q \in \mathbb{Z}_\infty^Q$ such that $e_q[q] = 0$ but $e_q[p] = \infty$ for $p \in Q \setminus \{q\}$. Let $p, q \in Q$. We consider the following descriptions of subsets of $\mathbb{Z}_\infty^Q \times \mathbb{Z}_\infty^Q$ which play the role of the compressed search space.

1. $T^{\mathrm{a}}_{p,q} \;=\; \Big\{(e_p, e_q) \cdot w \;\Big|\; w \in \Sigma^*\Big\}$
2. Let $T^{\mathrm{b}}_{p,q}$ be the least set which contains (e_p, e_q) and is closed such that for every $(x, y) \in T^{\mathrm{b}}_{p,q}$ and every $a \in \Sigma$, we have $(x, y) \cdot a \in T^{\mathrm{b}}_{p,q}$.

4 In the case $x \cdot \mu(u) = y \cdot \mu(u) = \infty^Q$ we can choose an arbitrary $m \in \mathbb{Z}$.

It is easy to verify that these two sets coincide. Indeed, $T^{\mathrm{a}}_{p,q} \subseteq T^{\mathrm{b}}_{p,q}$ follows from Lemma 2 with an inductive argument. For the base of induction, just observe $(e_p, e_q) \cdot \varepsilon = (e_p, e_q)$. To verify $T^{\mathrm{b}}_{p,q} \subseteq T^{\mathrm{a}}_{p,q}$ we use Lemma 2 to show that $T^{\mathrm{a}}_{p,q}$ has the closure property as in the definition of $T^{\mathrm{b}}_{p,q}$.

Consequently, we write $T_{p,q}$ for the set $T^{\mathrm{a}}_{p,q} = T^{\mathrm{b}}_{p,q}$.

Lemma 3. *Two states p, q of $\mathcal{A}$ are twins iff for every $(x, y) \in T_{p,q}$, $x[p] \in \mathbb{Z}$, $y[q] \in \mathbb{Z}$ imply $x[p] = y[q]$.*

Proof. $\Rightarrow$ Let $p, q \in Q$ be twins and $(x, y) \in T_{p,q}$ such that $x[p], y[q] \in \mathbb{Z}$. By the definition of $T^{\mathrm{a}}_{p,q}$, there are $w \in \Sigma^*$, $m \in \mathbb{Z}$ such that $(x, y) = (e_p, e_q) \cdot w =$

$$[\![(e_p \cdot \mu(w), e_q \cdot \mu(w))]\!] = \big(m \oplus e_p \cdot \mu(w),\ m \oplus e_q \cdot \mu(w)\big).$$

Hence, $x[p] = m + \mu(w)[p, p]$ and $y[q] = m + \mu(w)[q, q]$. By $x[p], y[q] \in \mathbb{Z}$, we observe $\mu(w)[p, p],\ \mu(w)[q, q] \in \mathbb{Z}$. Since p, q are twins, $\mu(w)[p, p] = \mu(w)[q, q]$ and the claim follows.

$\Leftarrow$ Let p, q be states of $\mathcal{A}$ and let $w \in \Sigma^*$ satisfying $\mu(w)[p, p],\ \mu(w)[q, q] \in \mathbb{Z}$. By the definition of $T^{\mathrm{a}}_{p,q}$, the pair $(x, y) = (e_p, e_q) \cdot w$ belongs to $T_{p,q}$. By arguing as for the other direction, there is some $m \in \mathbb{Z}$ such that $x[p] = m + \mu(w)[p, p]$ and $y[q] = m + \mu(w)[q, q]$. Hence, $x[p], y[q] \in \mathbb{Z}$, by assumption $x[p] = y[q]$ and it follows $\mu(w)[p, p] = \mu(w)[q, q]$. $\square$

For siblings p, q, let us call $(x, y) \in T_{p,q}$ a *critical pair for p, q* if $x[p] \in \mathbb{Z}$, $y[q] \in \mathbb{Z}$ but $x[p] \neq y[q]$. Lemma 3 says that siblings are not twins iff they admit a critical pair.

Lemma 4. *Assume that $\mathcal{A}$ satisfies the twins property. For every siblings p, q of $\mathcal{A}$, the entries of every pair in $T_{p,q}$ belong to $\{0, 1, \ldots, |Q|^2 \cdot \mathsf{dg}(\mathcal{A}), \infty\}$. In particular, $T_{p,q}$ is finite.*

Proof. Immediate conclusion from the definition of $T^{\mathrm{a}}_{p,q}$ and Lemma 1. $\square$

At this point, we can present a practical algorithm to decide the twins property.

At first, the algorithm constructs all pairs of siblings.

Then, it enlists *simultaneously* for every siblings p, q the set $T_{p,q}$ and searches for critical pairs. The definition of $T^{\mathrm{b}}_{p,q}$ gives such an algorithm which stops when $T_{p,q}$ is completely enlisted.

If the algorithm finds for some siblings p, q a critical pair, then it stops and says that $\mathcal{A}$ does not satisfy the twins property.

If all the processes to enlist sets $T_{p,q}$ terminate and the algorithm does not find a critical pair, then it stops and says that $\mathcal{A}$ satisfies the twins property.

The reader should be aware that even if critical pairs exist, all the sets $T_{p,q}$ might be finite.

If $\mathcal{A}$ does not satisfy the twins property, then Lemma 3 assures the existence of a critical pair. This pair is found, and the algorithm terminates and gives the correct answer. If $\mathcal{A}$ satisfies the twins property, then a critical pair cannot exist by Lemma 3. By Lemma 4, the sets $T_{p,q}$ are finite. Hence, the algorithm terminates and gives the correct answer.

Finally, we complete the proof of Theorem 2.

Proof (*Theorem 2, Decidability in PSPACE*)**.** We are quite fair and assume that $\mathcal{A} = [Q, \lambda, \mu, \varrho]$ is represented in a very compact way. Let us assume that Q is an initial sequence of the natural numbers, i.e., $Q = \{1, \ldots, |Q|\}$. Thus, Q might be represented by a binary number.

Moreover, we assume that only the non-∞ entries of λ, $\mu(a)$, and ϱ are represented. Thus, λ and ϱ could be represented by lists consisting of $Q \times \mathbb{Z}$-pairs. Similarly, for every $a \in \Sigma$, the matrix $\mu(a)$ is represented by a list of $Q \times Q \times \mathbb{Z}$-triples. We assume that all numbers in these lists are represented binary.

Let us assume that for every $q \in Q$, we have $\lambda[q] \in \mathbb{Z}$ or there are $a \in \Sigma$, $p \in Q$ satisfying $\mu(a)[p, q] \in \mathbb{Z}$.[5] Consequently, $|Q|$ is at most linear in the size of $\mathcal{A}$, since for every $q \in Q$, there is at least one entry in the lists representing λ, $\mu(a)$, and ϱ.

We present a non-deterministic PSPACE-algorithm which admits a successful run iff the input WFA does not satisfy the twins property. By Savitch's determinization theorem, the twins property is decidable in PSPACE.

The algorithm has a counter and a pair of arrays (x, y) to store a member of some set $T_{p,q}$. Initially, the algorithm guesses a pair of siblings p, q, sets $(x, y) := (e_p, e_q)$ and the counter to 0.

The algorithm guesses some $a \in \Sigma$ and sets $(x, y) := (x, y) \cdot a$. Then, it checks whether (x, y) includes an integer entry strictly larger than $|Q|^2 \cdot \mathsf{dg}(\mathcal{A})$ or whether (x, y) is a critical pair. If so, it stops successfully. Otherwise, the algorithm increments the counter. If the counter is bigger than

$$\big(|Q|^2 \cdot \mathsf{dg}(\mathcal{A}) + 2\big)^{2|Q|}, \tag{3}$$

then the computation terminates and fails. Otherwise, the algorithm loops back and guesses some $a \in \Sigma$ and so on… Note that the bound in (3) is the number pairs in $\mathbb{Z}^Q_\infty \times \mathbb{Z}^Q_\infty$ whose entries belong to $0, \ldots, |Q|^2 \cdot \mathsf{dg}(\mathcal{A}), \infty$.

The algorithm obviously terminates. If there is some successful run, then $\mathcal{A}$ does not satisfy the twins property either by Lemma 4 (in contraposition) or by Lemma 3.

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5 If $\mathcal{A}$ does not satisfy this requirement, then we can preprocess $\mathcal{A}$ in polynomial time and remove all these obscure states.

Conversely, assume that $\mathcal{A}$ does not satisfy the twins property. By Lemma 3, there is a critical pair in $T_{p,q}$ for some siblings p and q. One run of the algorithm works directly towards this critical pair. In particular, this run does not involve some pair (x, y) twice. By a straightforward counting argument, either the run reaches the critical pair, or the run achieves a pair (x, y) with an integer entry strictly larger than $|Q|^2 \cdot \mathsf{dg}\,(\mathcal{A})$.

Guessing siblings and computing $\mathsf{dg}\,(\mathcal{A})$ is certainly possible in PSPACE. For the counter, we need roughly

$$2|Q| \cdot \big(2\mathsf{ld}(|Q|) + \mathsf{ld}(\mathsf{dg}\,(\mathcal{A}))\big) \tag{4}$$

bits. Clearly, $\mathsf{ld}(\mathsf{dg}\,(\mathcal{A}))$ is essentially the size of some entry of some matrix $\mu(a)$, i.e., part of the input. Hence, $\mathsf{ld}(\mathsf{dg}\,(\mathcal{A}))$ is linear in the size of $\mathcal{A}$, and thus, (4) is polynomial in the size of $\mathcal{A}$.

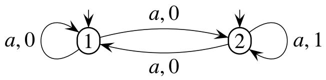
For (x, y), we need space for $2|Q|$ entries which range over $0, \ldots, |Q|^2 \cdot \mathsf{dg}\,(\mathcal{A}), \infty$, which is roughly the same as (4). To compute $(x, y) \cdot a$ and to check whether (x, y) is a critical pair, we need an additional variable of the same size as (x, y), integer variables, and a loop variables which range over Q. However, the storage size for these variables is less than (4).

To sum up, the space is polynomial in the size of $\mathcal{A}$. $\square$

5. Some interesting examples

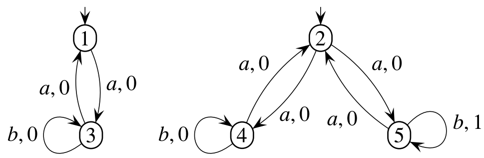
Lemmas 3 and 4 may lead us to the conjecture that for siblings p, q the set $T_{p,q}$ is finite iff p, q are twins. However, both directions of this conjecture are false.

Example 1. Let $\Sigma = \{a\}$ and $\mathcal{A}$ be the following WFA:



The states 1 and 2 are siblings because they are initial states. They are not twins because $\mu(a)[1, 1] = 0$ but $\mu(a)[2, 2] = 1$. For $w \in \Sigma^*$ satisfying $|w| \geq 2$, we have $\mu(w) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. By the definition of $T^{a}_{p,q}$, the set $T_{p,q}$ is finite for every $p, q \in \{1, 2\}$. $\square$

Example 2. Let $\Sigma = \{a, b\}$ and $\mathcal{A}$ be the following WFA:



The states 1 and 2 are siblings because they are initial states. Moreover, 1 and 2 are twins: For words $w \in (ab^*a)^*$, we have $\mu(w)[1, 1] = \mu(w)[2, 2] = 0$. For any other word w, we have $\mu(w)[1, 1] = \mu(w)[2, 2] = \infty$.

However, $T_{1,2}$ is infinite: for every n, we have

$$(e_1, e_2) \cdot ab^n = \big((\infty, \infty, 0, \infty, \infty),\ (\infty, \infty, \infty, 0, n)\big).$$

There are critical pairs in $T_{3,5}$ and $T_{4,5}$. $\square$

6. Discussion

As a main conclusion, the twins property is a decidable sufficient condition for the termination of Mohri's algorithm, and hence, it is a decidable sufficient condition for the existence of a deterministic equivalent. The paper may have an impact on the determinization problem of WFA. It is quite surprising that the twins property is rather easy to decide, in particular in contrast to the results in [17,18] which utilize finite semigroup theory.

It remains open how the ideas of the paper may be generalized to other semirings, (e.g. min-semirings [19]) or trees.

Acknowledgements

The author thanks the anonymous referee for his useful remarks.

Supported by the research project "Descriptive Complexity of Small Complexity Classes" of the German Research Foundation.

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